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CONDENSATIONAL GROWTH OF DROPLETS

An analysis of vapour condensation on droplets is presented. General approximate expressions for the parameters of the droplet mass distribution function are derived from the analytical solution. These expressions are valid for a broad class of droplet mass growth rates and arbitrary initial distributions. The accuracy of these expressions is investigated and the errors are found to be small. The method of moments for log-normal and gamma distributions is applied to the problem of vapour condensation. Some approximate formulas for parameters of these distributions are derived and their accuracy is investigated.

Nomenclature

e	– relative error
g	– condensational mass growth rate, [s ⁻¹]
m	– mass, [kg]
M	– total mass of droplets per unit volume, [kg · m ⁻³]
n	– droplet mass distribution function, [kg ⁻¹ · m ⁻³]
N	– total number of droplets per unit volume, [m ⁻³]
t	– time, [s]

Subscripts

max	– the value of mass at which droplet mass distribution function attains maximum
0	– the initial value

INTRODUCTION

The problem of condensation of vapour on droplets or solid aerosol particles is important in many different areas, including atmospheric physics [8], [9] and nuclear reactor safety [10], [11].

Usually we are interested in the evolution of the droplet mass distribution function when the condensational mass growth rate of a droplet and the initial distribution are known.

Some exact analytical solutions obtained before [5] are used for derivation of useful approximate formulas and their accuracy is investigated.

Two initial distributions are considered, namely log-normal and gamma distributions with different parameters. The method of moments is used for investigation of these parameters and some approximate formulas are derived.

The investigations presented here are illustrated by some numerical examples.

1. THEORY

The balance equation for the condensation of a vapour on droplets with given initial condition has the form

$$\frac{\partial n(m, t)}{\partial t} = - \frac{\partial}{\partial m} [g(m, t) n(m, t)] \quad (1)$$

$$n(m, 0) = n_0(m) \quad (2)$$

where $n(m, t)$ is the droplet mass distribution function (defined so that $n(m, t)dm$ is the average number of droplets per unit volume with masses between m and $m + dm$ at time t), $g(m, t)$ is the condensational mass growth rate of a droplet and $n_0(m)$ is the initial droplet mass distribution function.

In most cases of practical importance we may assume that

$$g(m, t) = h(t)f(m) \quad (3)$$

The solution of the initial value problem (1), (2) can be obtained by the method of characteristics [5], [11]. When the assumption (3) is valid, it has the form

$$F(m) - H(t) = F(\tau) \quad (4)$$

$$n(m, t) = \frac{f(\tau)}{f(m)} n_0(\tau) \quad (5)$$

where

$$F(m) = \int_0^m \frac{dm'}{f(m')} \quad (6)$$

$$H(t) = \int_0^t h(t') dt' \quad (7)$$

It follows from physical considerations that the functions $h(t)$ and $f(m)$ are real positive functions of real positive arguments. Hence, $F(m)$ and $H(t)$ are real positive increasing functions of m and t , respectively. Obviously, $F(0) = H(0) = 0$.

Since n_0 is defined for nonnegative values of argument, then it follows that $\tau \geq 0$. Moreover, from eq. (4) it follows that $m(\tau, t) > \tau$ for any t and that $m(\tau, t)$ is an increasing function of time.

If $n_0(m) = 0$ for $m < m_0$, then the distribution $n(m, t)$ moves to the right along m axis, when time increases and $m_0(t)$ is an increasing function of time (obviously m_0 can be equal to zero and really it is in most cases).

Similarly, $m_*(t)$ increases with time for any $m_*(0) > m_0$.

If $m_2(0) > m_1(0) > m_0$, then

$$\Delta m_{12}(t) = m_2(t) - m_1(t) \quad (8)$$

increases with time. When the initial distribution $n_0(m)$ attains its maximum at the value m_{\max} , then $n(m, t)$ has the maximum at the value $m_{\max}(t)$.

It can easily be seen that

$$\Delta m_r(t) = \frac{\Delta m_{12}(t)}{m_{\max}(t)} \quad (9)$$

decreases with time.

From eq. (5) it follows that

$$n_{\max}(t) = n[m_{\max}(t), t] = \frac{f(\tau = m_{\max})}{f[m(\tau = m_{\max})]} n_0(\tau = m_{\max})$$

and since $m_{\max}(t)$ and $f(m)$ are increasing functions, then

$$f(\tau = m_{\max}) < f[m(\tau = m_{\max})] \text{ and } n_{\max}(t) < n_0(m_{\max})$$

Therefore, the ratio R

$$R(t) = \frac{n_{\max}(t)}{n_0(m_{\max})} \quad (10)$$

is less than unity.

It follows from eqs. (4) and (6) that

$$\frac{dm}{d\tau} = \frac{f(m)}{f(\tau)}$$

Therefore, it results from eq. (5) that

$$\int_{m_1(t)}^{m_2(t)} n(m, t) dm = \int_{m_1}^{m_2} n_0(\tau) d\tau \quad (11)$$

for any $m_2 > m_1 \geq m_0$.

One can notice that, in particular, the number of droplets with masses lying in the interval $[m_0(t), m_{\max}(t)]$ does not change with time and is the same as for the initial distribution.

Some exact analytical solutions and useful approximations can be obtained when

$$f(m) = a m^\alpha$$

It results from the research of Barrett and Clement [1], [2] that the above formula is valid for the case, when radiative heat flux from droplets is neglected. In a free molecule range $\alpha = 2/3$ and in a continuum range $\alpha = 1/3$.

From eqs. (4)-(7) it follows [5] that

$$m(\tau, t) = \left[\tau^{1-\alpha} + (1-\alpha) a H(t) \right]^{\frac{1}{1-\alpha}} \quad (12)$$

and

$$m_i(t) = m(\tau = m_i, t) = \left[m_i^{1-\alpha} + (1-\alpha) a H(t) \right]^{\frac{1}{1-\alpha}} \quad (13)$$

for any $m_i \geq m_0$.

The ratio $R(t)$ can be expressed in the form

$$R(t) = \frac{m_{\max}^\alpha}{\left[m_{\max}^{1-\alpha} + (1-\alpha) a H(t) \right]^{\frac{1}{1-\alpha}}} \quad (14)$$

It is easily seen from the above considerations that the total mass of droplets per unit volume $M(t)$ can be expressed in the form [5]

$$M(t) = \int_0^\infty m n(m, t) dm = \int_0^\infty m(\tau, t) n_0(\tau) d\tau \quad (15)$$

Some useful approximations can be made when

$$\frac{m_i^{1-\alpha}}{(1-\alpha) a H(t)} \ll 1$$

Taking into account only first two terms in the series expansion of $m_i(t)$, one gets the approximate expressions

$$m_i(t) \approx \tilde{m}_i(t) = [(1 - \alpha) a H(t)]^{\frac{1}{1-\alpha}} \left[1 + \frac{m_i^{1-\alpha}}{(1 - \alpha)^2 a H(t)} \right] \quad (13^*)$$

$$\begin{aligned} \Delta m_{12}(t) &\approx \Delta \tilde{m}_{12}(t) = \\ &= \frac{1}{(1 - \alpha)} [(1 - \alpha) a H(t)]^{\frac{\alpha}{1-\alpha}} \left(m_2^{1-\alpha} - m_1^{1-\alpha} \right) \end{aligned} \quad (8^*)$$

$$\begin{aligned} \Delta m_r(t) &\approx \Delta \tilde{m}_r(t) = \\ &= \frac{1}{(1 - \alpha)} [(1 - \alpha) a H(t)]^{-1} \left(m_2^{1-\alpha} - m_1^{1-\alpha} \right) \end{aligned} \quad (9^*)$$

$$R(t) \approx \tilde{R}(t) = m_{\max}^{\alpha} [(1 - \alpha) a H(t)]^{-\frac{\alpha}{1-\alpha}} \quad (14^*)$$

When $n_0(\tau) \neq 0$ only for sufficiently small values of τ , then

$$\begin{aligned} M(t) &= \int_0^{\infty} m n(m, t) dm \approx \\ &\approx \tilde{M}(t) = m(0, t) \int_0^{\infty} n_0(\tau) d\tau = m(0, t) N_0 \end{aligned}$$

where M is the total mass of droplets per unit volume and N is the total number of droplets per unit volume (obviously $N(t) = N_0$).

From eq. (12) it follows that

$$M(t) \approx \tilde{M}(t) = [(1 - \alpha) a H(t)]^{\frac{1}{1-\alpha}} N_0 \quad (16)$$

The considerations presented above are quite general and do not require the knowledge of the initial distribution. However, more information about the droplet mass distribution function can be obtained when the initial distribution is specified.

Two distributions are especially interesting, namely log-normal and gamma distributions.

The method of moments gives the possibility to obtain quite simply approximate solutions with good accuracy [6], [7], [10], [11].

a) the log-normal distribution is given by

$$n(m) = \frac{N}{(2\pi\sigma)^{1/2}} \frac{1}{m} \exp \left[-\frac{1}{2\sigma} \ln^2 \left(\frac{m}{m_0} \right) \right] \quad (17)$$

and is characterized by three parameters: N , m_0 and σ . The set of equations for parameters has the form

$$\frac{dm_0}{dt} = am_0^\alpha e^{-(1-\alpha^2)\sigma/2} (2 - e^{-(1-\alpha)\sigma}) \quad (18)$$

$$\frac{d\sigma}{dt} = 2am_0^{\alpha-1} e^{-(1-\alpha^2)\sigma/2} (e^{-(1-\alpha)\sigma} - 1) \quad (19)$$

There is no equation for N , since $N(t) = \text{const}$. The initial conditions have the form

$$m_0(0) = m_{00}, \quad \sigma(0) = \sigma_0 \quad (20)$$

Dividing eqs. (18), (19) side by side, one gets the equation

$$\frac{dm_0}{d\sigma} = \frac{1}{2} m_0 \frac{2 - e^{-(1-\alpha)\sigma}}{e^{-(1-\alpha)\sigma} - 1}$$

which has the solution, satisfying the initial conditions, in the form

$$m_0 = m_{00} e^{\sigma_0} \left[1 - e^{-(1-\alpha)\sigma_0} \right]^{\frac{1}{2(1-\alpha)}} e^{-\sigma} \left[1 - e^{-(1-\alpha)\sigma} \right]^{\frac{1}{2(1-\alpha)}} \quad (21)$$

The substitution of m_0 into eq. (19) gives the equation

$$\frac{d\sigma}{dt} = -2am_{00}^{-(1-\alpha)} e^{-(1-\alpha)\sigma_0} \left(1 - e^{-(1-\alpha)\sigma_0} \right)^{-\frac{1}{2}} e^{(1-\alpha)^2\sigma/2} \left(1 - e^{-(1-\alpha)\sigma} \right)^{\frac{3}{2}} \quad (22)$$

This equation can be solved analytically when one assumes that $(1-\alpha)^2 \approx 1-\alpha$, i.e. when $\alpha \ll 1$. The approximate solution has the form

$$\sigma \approx \tilde{\sigma} = \frac{1}{1-\alpha} \ln \left(1 + \frac{A}{t^2} \right) \quad (23)$$

where

$$A = [(1-\alpha)a]^{-2} m_{00}^{2(1-\alpha)} e^{2(1-\alpha)\sigma_0} \left(1 - e^{-(1-\alpha)\sigma_0} \right) \quad (24)$$

Since $A/t^2 \ll 1$, then

$$\bar{\sigma} \approx \frac{1}{1-\alpha} \frac{A}{t^2} \quad (23^*)$$

Taking into account that $\sigma \ll 1$, one gets from eq. (21)

$$m_0(t) \approx \tilde{m}_0(t) = [(1-\alpha)at]^{\frac{1}{1-\alpha}} \quad (25)$$

$$m_{\max}(t) \approx \tilde{m}_{\max}(t) = [(1-\alpha)at]^{\frac{1}{1-\alpha}} \quad (26)$$

For the log-normal distribution

$$M = Nm_0 e^{\sigma/2}$$

then

$$M(t) \approx \tilde{M}(t) = [(1-\alpha)at]^{\frac{1}{1-\alpha}} N_0 \quad (16^*)$$

b) the gamma distribution is given by

$$n(m) = \frac{N}{\Gamma(v+1)} \frac{1}{m_0} \left(\frac{m}{m_0}\right)^v \exp\left(-\frac{m}{m_0}\right) \quad (27)$$

The set of equations for parameters has the form

$$\frac{dm_0}{dt} = (2\alpha - 1) a m_0^\alpha \frac{1}{v+1} \frac{\Gamma(\alpha + v + 1)}{\Gamma(v+1)} \quad (28)$$

$$\frac{dv}{dt} = 2(1-\alpha) a m_0^{\alpha-1} \frac{\Gamma(\alpha + v + 1)}{\Gamma(v+1)} \quad (29)$$

The initial conditions have the form

$$m_0(0) = m_{00}, \quad v(0) = v_0 \quad (30)$$

Dividing the above equations side by side and solving the resulting equation with initial conditions, one gets

$$m_0 = m_{00} (v_0 + 1)^{\frac{1-2\alpha}{2(1-\alpha)}} (v + 1)^{-\frac{1-2\alpha}{2(1-\alpha)}} \quad (31)$$

The substitution of m_0 into eq. (28) gives the equation

$$\frac{dv}{dt} = 2(1-\alpha) a m_0^{\alpha-1} (v_0 + 1)^{\alpha-\frac{1}{2}} (v + 1)^{-\alpha+\frac{1}{2}} \frac{\Gamma(\alpha + v + 1)}{\Gamma(v+1)} \quad (32)$$

Taking only the first term in Stirling asymptotic expansion of gamma function and assuming that $\alpha < \nu + 1$, one gets

$$\frac{\Gamma(\alpha + \nu + 1)}{\Gamma(\nu + 1)} \approx (\nu + 1)^\alpha$$

Then it is possible to obtain approximate analytical solution of eq. (32) in the form

$$\nu(t) \approx \tilde{\nu}(t) = (1 - \alpha)^2 a^2 m_{00}^{-2(1-\alpha)} (\nu_0 + 1)^{-(1-2\alpha)} t^2 \quad (33)$$

and

$$m_0(t) \approx \tilde{m}_0(t) = m_{00}^{2(1-\alpha)} (\nu_0 + 1)^{1-2\alpha} \left[(1 - \alpha) a t \right]^{\frac{1-2\alpha}{1-\alpha}} \quad (34)$$

$$m_{\max}(t) \approx \tilde{m}_{\max}(t) = \tilde{m}_0(t) \tilde{\nu}(t) = \left[(1 - \alpha) a t \right]^{\frac{1}{1-\alpha}} \quad (26^*)$$

$$M(t) \approx \tilde{M}(t) = \left[(1 - \alpha) a t \right]^{\frac{1}{1-\alpha}} N_0 \quad (16^{**})$$

2. NUMERICAL ANALYSIS

The presented considerations are illustrated below by some numerical examples.

The calculations are performed for the data given below

a) $h(t) = 1$, hence $H(t) = t$,

b) $f(m) = a m^\alpha$, where $\alpha = 1/3$ and $a = 0.27076 \cdot 10^{-8} \text{ kg}^{2/3} \cdot \text{s}^{-1}$.

The value of a has been used before [5] and had been obtained from Barrett and Clement formula for a continuum range [1] for the following data: temperature 100°C , air partial pressure 0.1 MPa, vapour saturation $S = 1.01$, vapour molecule sticking probability $S_A = 0.04$ and accommodation coefficients $\alpha_g = \alpha_v = 1.0$.

1. Some general formulas for characteristics parameters of $n(m, t)$, valid for any initial distribution $n_0(m)$, are derived above. Also some approximations of these parameters have been obtained.

It is assumed that the initial distribution starts at $m = 0$ (i.e. $m_0 = 0$) and attains maximum at the value $m = m_{\max}$. The values of $m_{\max}(t)$ are calculated from eq. (13) and the approximate values of $\tilde{m}_{\max}(t)$ from eq. (13*).

We introduce the relative error of the quantity X defined as follows

$$e(X) = \left| \frac{\tilde{X}(t) - X(t)}{X(t)} \right| 100\% \quad (34)$$

where X and \tilde{X} denote the exact and approximate values, respectively.

The error $e(m_{\max})$ is shown in Fig.1 as a function of time for $m_{\max} = 10^{-21}$ kg, 10^{-18} kg and 10^{-15} kg.

The error $e(R)$ versus time is shown in Fig.2 for $m_{\max} = 10^{-21}$ kg, 10^{-18} kg and 10^{-15} kg.

The calculations of error $e(\Delta m_r)$ are performed for $m_1 = 0$ and $m_2 = m_{\max}$, where $m_{\max} = 10^{-21}$ kg, 10^{-18} kg and 10^{-15} kg; $\Delta m_r(t)$ and $\Delta \tilde{m}_r(t)$ are calculated from eq. (13) and eq. (9*), respectively.

The error $e(\Delta m_r)$ versus time is shown in Fig.3.

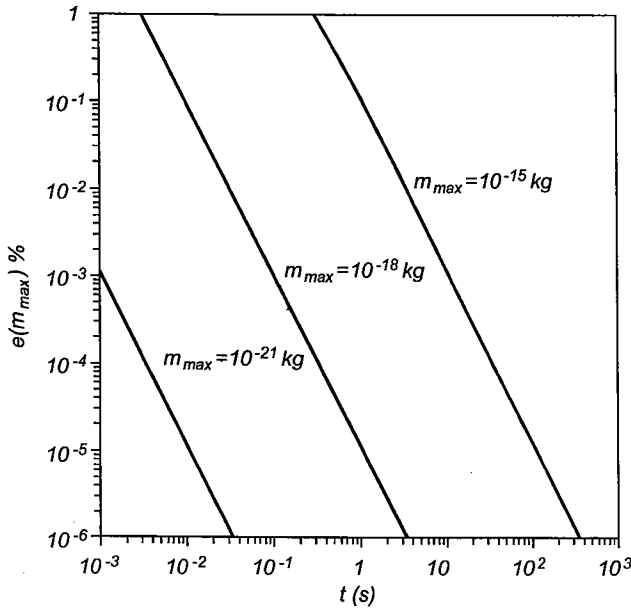


Fig. 1. Error $e(m_{\max})$ versus time

2. Two initial distributions are considered, namely log-normal and gamma distribution, with $m_{00} = 10^{-21}$ kg, 10^{-18} kg and 10^{-15} kg. The values of $M(t)$ are calculated from eq. (15), and the values of $\tilde{M}(t)$ are calculated from eq. (16).

The error $e(M)$ for gamma distribution with $v_0 = 1$ and $v_0 = 3$ is shown in Fig.4, and log-normal distribution with $\sigma_0 = 1$ and $\sigma_0 = 3$ is shown in Fig.5.

The parameters v and σ as functions of time, for $m_{00} = 10^{-21}$ kg and $\sigma_0 = v_0 = 3$, are shown in Fig.6.

The errors of m_0 , v and σ , for the same values, are shown in Fig.7 (the errors are defined as above).

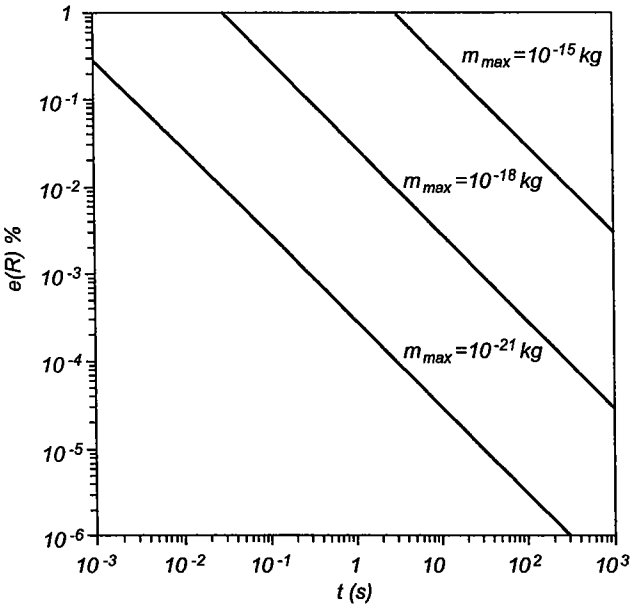


Fig. 2. Error $e(R)$ versus time

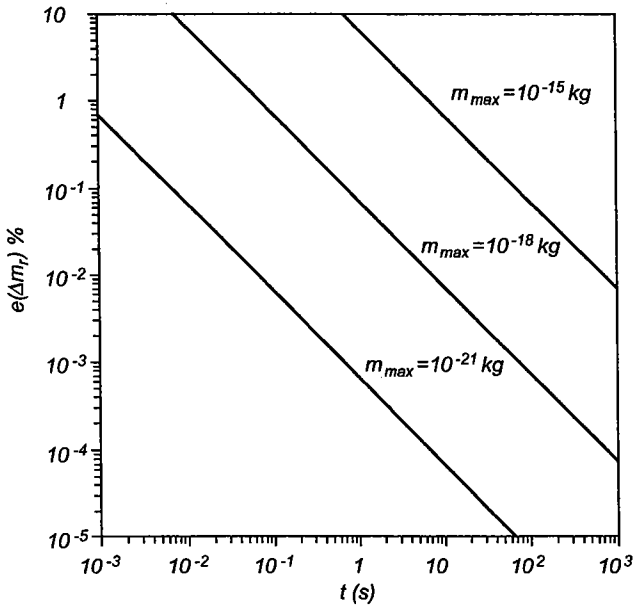


Fig. 3. Error $e(\Delta m_r)$ versus time

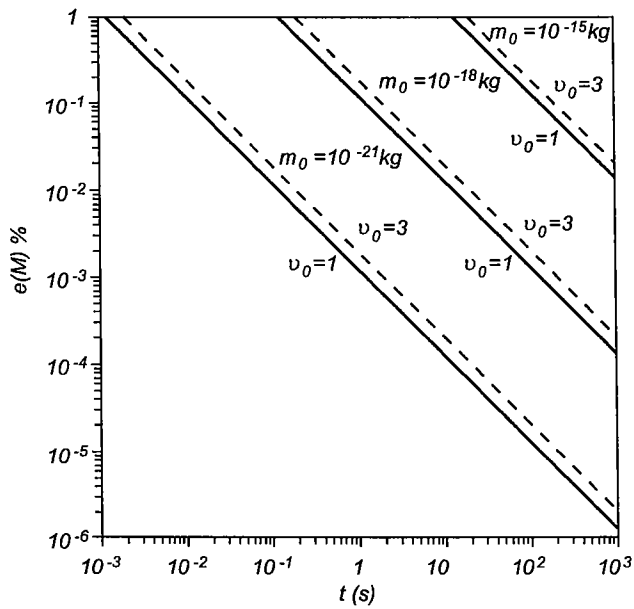


Fig. 4. Error $e(M)$ versus time for gamma distribution

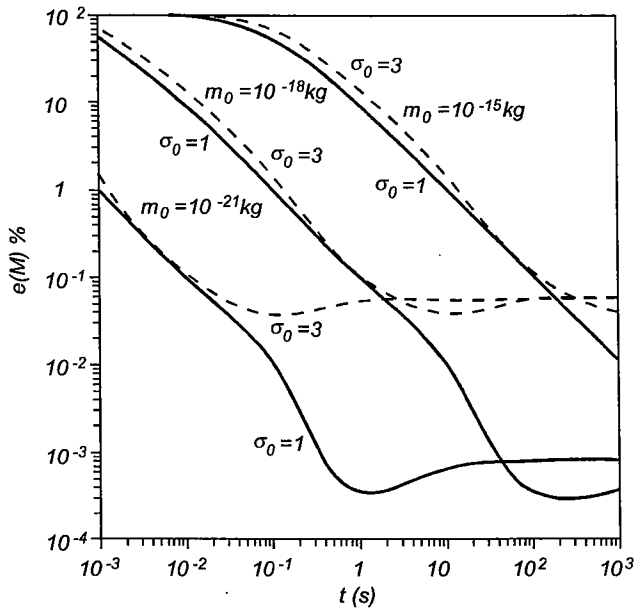


Fig. 5. Error $e(M)$ versus time for log-normal distribution

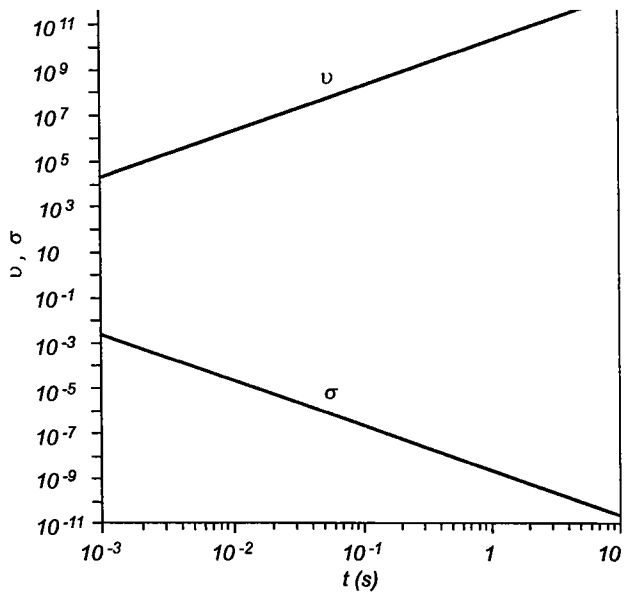


Fig. 6. σ and v versus time for $m_{00} = 10^{-21}$ kg and $\sigma_0 = v_0 = 3$

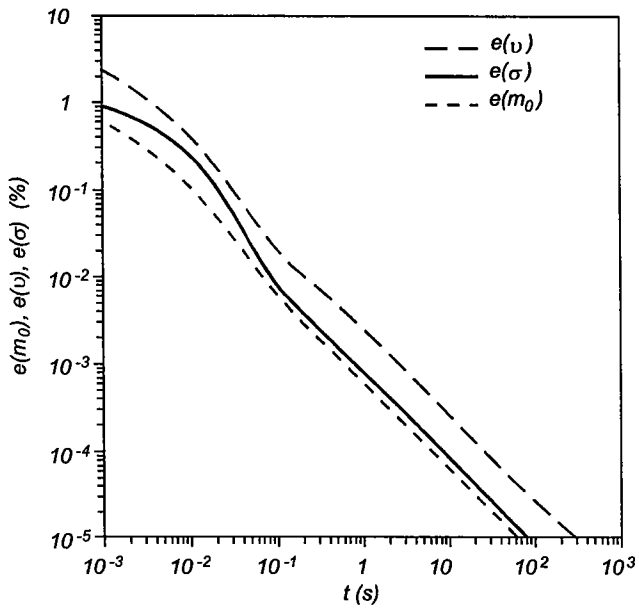


Fig. 7. Errors $e(m_0)$, $e(v)$ and $e(\sigma)$ versus time for $m_{00} = 10^{-21}$ kg, and $\sigma_0 = v_0 = 3$

CONCLUSIONS

The investigation of the droplet mass distribution function has been performed and illustrative calculations have been carried out.

Some general information concerning the time evolution of m_{\max} , Δm_r and the integral of the droplet mass distribution function can be obtained for arbitrary mass growth rates.

When for the given condensational mass growth rate the exact analytical solution exists, then some approximate formulas for the above-mentioned quantities can be derived.

When the initial distribution is either log-normal or gamma distribution, then it is possible to use the method of moments for the determination of the approximate solution. The approximate formulas for parameters of these distributions ensure good accuracy for sufficiently large values of time.

The approximate formulas for m_{\max} and M are identical with those obtained from exact analytical solutions.

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KONDENSACYJNY WZROST KROPEL

Streszczenie

W pracy prezentowana jest analiza kondensacji pary na kropkach. Ogólne przybliżone wyrażenia dla parametrów funkcji rozkładu mas kropeł wyprowadza się z rozwiązania analitycznego. Wyrażenia te są słuszne dla szerokiej klasy prędkości wzrostu mas i dowolnych rozkładów początkowych. Badana jest dokładność tych wyrażeń i stwierdza się, że błędy są małe. Do zagadnienia kondensacji stosowana jest metoda momentów dla rozkładu logarytmiczno-normalnego i rozkładu gamma. Wyprowadzane są przybliżone wzory dla parametrów tych rozkładów i badana jest ich dokładność.

КОНДЕНСАЦИОННЫЙ РОСТ КАПЕЛЬ

Краткое содержание

Приводится анализ конденсации пара на каплях. Общие приближенные формулы для параметров функции распределения масс капель выводятся из аналитического решения. Эти формулы пригодны для широкого класса скорости роста массы капель и для любых начальных распределений. Приводится анализ точности этих формул и обнаруживается, что погрешности малые. Метод моментов применяется для решения проблемы конденсации пара для логарифмически-нормального и гамма распределений.

Выводятся приближенные формулы для параметров этих распределений и исследуется их точность.